MATH 127 - Midterm Exam I - Review

Fall 2021 - Sections 12.1-12.7, 14.1-14.7

Midterm Exam 1: Wednesday 9/29, 5:50-7:50 pm

The following is a list of important concepts will be tested on Midterm Exam 1. This is not a complete list of the material that you should know for the course, but the review provides a summary of concepts and the problems are a good indication of what will be emphasized on the free response portion of the exam. A thorough understanding of all of the following concepts will help you perform well on the exam. Some places to find problems on these topics are the following: in the book, in the slides, in the homework, on quizzes, and Achieve.

Vector Geometry and Quadric Surfaces: (Sections 12.1 - 12.6)

A vector $\vec{r}$ can be described using component notation $\langle a, b, c\rangle$ or standard basis notation $a\vec{i} + b\vec{j} + c\vec{k}$.

A vector has a magnitude, its length, $|\vec{r}| = \sqrt{a^2 + b^2 + c^2}$.

Suppose that $\vec{a} = \langle a_1, a_2, a_3\rangle$ and $\vec{b} = \langle b_1, b_2, b_3\rangle$; let c be a scalar.

(i) **Scalar Multiplication:** A scalar multiplied with a vector resulting in a vector. Scalar multiplication changes the magnitude of a vector, not it's direction. $c\vec{a} = \langle ca_1, ca_2, ca_3\rangle$

(ii) **Vector Addition:** Two vectors are added to create a vector. Visually, vectors are added through the Parallelogram or Triangle Law. $\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$

(iii) **Dot Product:** Two vectors are multiplied to create a scalar. Work is an example of the dot product. If θ is the angle between $\vec{a}$ and $\vec{b}$, then

$$\vec{a} \cdot \vec{b} = |\vec{a}| \left|\vec{b}\right| \cos(\theta) = a_1b_1 + a_2b_2 + a_3b_3$$

(iv) **Cross Product:** Two vectors are multiplied to create a vector. $a \vec{\times} b$ is orthogonal to both $\vec{a}$ and $\vec{b}$. Torque is an example of the cross product. The determinant of a 2×2 matrix is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, this observation is used in computing the cross product

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle$$

Important Identities:

(i) $\vec{v} \cdot \vec{v} = |\vec{v}|^2$

(ii) $\vec{v} \perp \vec{u}$ if and only if $\vec{v} \cdot \vec{u} = 0$.

(iii) Parallelogram formed by $\vec{u}$, $\vec{v}$ has area $|\vec{u} \times \vec{v}|$.

(iv) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.

Lines

Given a point $P(x_0, y_0, z_0)$ on the line and a directional vector $\vec{v} = \langle a, b, c \rangle$:

$$\frac{\text{Vector Equation}}{\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t\,\langle a, b, c \rangle}$$

Two lines are either parallel (parallel direction vectors), intersecting, or skew.

Planes

Given a point $P(x_0, y_0, z_0)$ on the plane and a normal vector $\vec{n} = \langle a, b, c \rangle$:

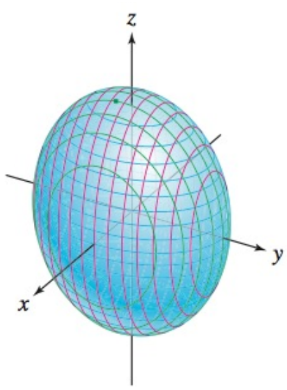
$$\frac{\text{Scalar Equation}}{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

Two planes are parallel (parallel normal vectors) or intersect along a line.

Quadric Surfaces:

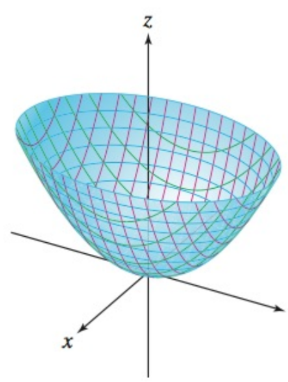
Ellipsoid

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$



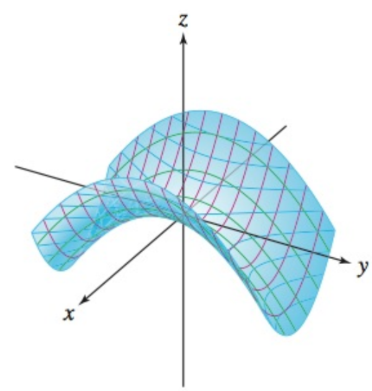
Elliptic Paraboloid

$$z = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$$



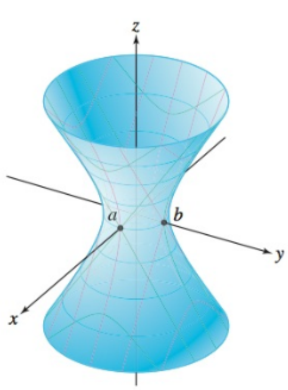
Hyperbolic Paraboloid

$$z = \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$$



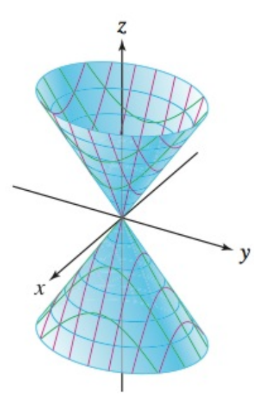
Hyperboloid of 1-Sheet

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 1$$



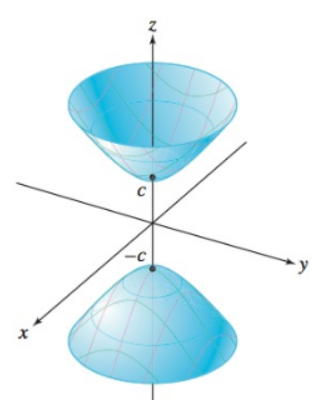
Cone

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 0$$



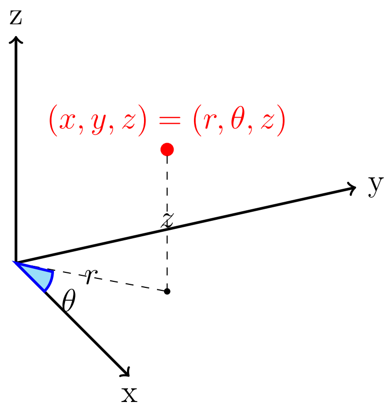
Hyperboloid of 2-Sheets

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = -1$$



Cylindrical and Spherical Coordinates:

Cylindrical Coordinates:



To convert from cylindrical coordinates to Cartesian coordinates:

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad z = z$$

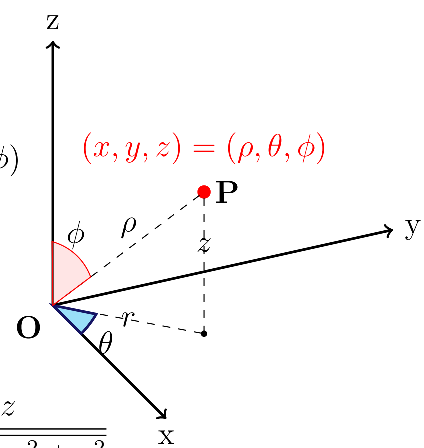
To convert from Cartesian coordinates to cylindrical coordinates:

$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x} \qquad z = z$$

Spherical Coordinates:

Conversion from spherical to Cartesian:

$$x = \rho\sin(\phi)\cos(\theta) \qquad y = \rho\sin(\phi)\sin(\theta) \qquad z = \rho\cos(\phi)$$

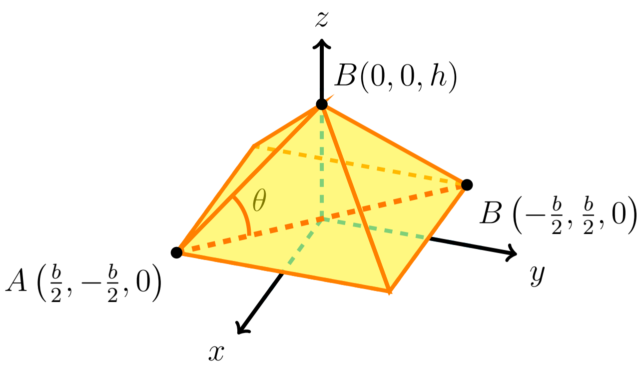


Conversion from Cartesian to spherical:

$$\rho = \sqrt{x^2 + y^2 + z^2} \qquad \tan(\theta) = \frac{y}{x} \qquad \cos(\phi) = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

Exercises:

1. Find the angle between the side of a square pyramid and a diagonal on the base.



We are searching for the angle formed by $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} = \left\langle \frac{-b}{2}, \frac{b}{2}, h \right\rangle \qquad \overrightarrow{AC} = \langle -b, b, 0 \rangle$$

Using the definition of the dot product:

$$\vec{v} \cdot \vec{u} = |\vec{v}|\ |\vec{u}| \cos(\theta)$$

$$\theta = \arccos\left(\frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\ \left|\overrightarrow{AC}\right|} \right) = \arccos\left(\frac{b^2}{\sqrt{0.5b^2 + h^2}\,\sqrt{2b^2}} \right) = \arccos\left(\frac{b}{\sqrt{b^2 + 2h^2}} \right).$$

2. If $|\vec{u}| = 2$, $|\vec{v}| = 3$, and the angle between the vectors is $30°$, then find $(\vec{u}+\vec{v}) \cdot (\vec{u}-\vec{v})$.

Recall that $\vec{x} \cdot \vec{x} = |\vec{x}|^2$ and $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$.

Using distribution of the dot product:

$$(\vec{u}+\vec{v}) \cdot (\vec{u}-\vec{v}) = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{u} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{v} = |\vec{u}|^2 - |\vec{v}|^2 = 2^2 - 3^2 = -5$$

3. Find two unit vectors orthogonal to both $\vec{a} = \langle 3, 1, 1 \rangle$ and $\vec{b} = \langle -1, 2, 1 \rangle$.

The two unity vectors are $\dfrac{\vec{n}}{|\vec{n}|}$ and $\dfrac{-\vec{n}}{|\vec{n}|}$ where $\vec{n} = \vec{a} \times \vec{b}$.

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 1 \\ -1 & 2 & 1 \end{vmatrix} = \langle -1, -4, 7 \rangle$$

Since $\vec{n} = \sqrt{66}$, the vectors are $\left\langle \frac{-1}{\sqrt{66}}, \frac{-4}{\sqrt{66}}, \frac{7}{\sqrt{66}} \right\rangle$ and $\left\langle \frac{1}{\sqrt{66}}, \frac{4}{\sqrt{66}}, \frac{-7}{\sqrt{66}} \right\rangle$.

4. Identify and sketch the graph of each surface:

(A) $y^2 + z^2 = 1 + x^2$

Hyperboloid of 1-Sheet, centered at $(1, 0, 0)$ opening along the x-axis.

(B) $-x^2 + y^2 - 4z^2 = 4$

Hyperboloid of 2-Sheets, with vertices at $(0, \pm 2, 0)$ opening along the y-axis.

(C) $4x^2 + 4y^2 - 8y + z^2 = 0$

$x^2 + (y-1)^2 + \frac{z^2}{2^2} = 1$ Ellipsoid, centered at $(0, 1, 0)$.

(D) $x = y^2 + z^2 - 2y - 4z + 5$

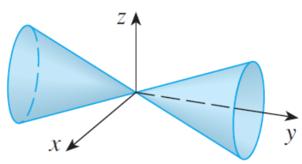
$x = (y - 1)^2 + (z - 2)^2$ Elliptic Paraboloid, centered at $(0, 1, 2)$ opening positively on x-axis.

5. Match the following equations with the graphs:

(A) $x^2 + 4y^2 + 9z^2 = 1$ (C) $x^2 - y^2 + z^2 = 1$ (E) $y = 2x^2 + z^2$ (G) $x^2 + 2z^2 = 1$

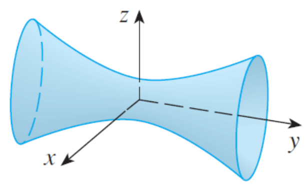
(B) $9x^2 + y^2 + z^2 = 1$ (D) $-x^2 + y^2 - z^2 = 1$ (F) $y^2 = x^2 + 2z^2$ (H) $y = x^2 - z^2$

I



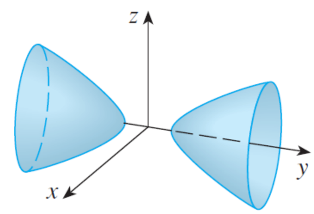
(F) $y^2 = x^2 + 2z^2$

II



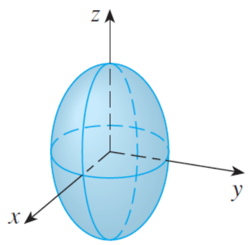
(C) $x^2 - y^2 + z^2 = 1$

III



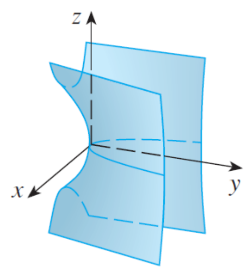
(D) $-x^2 + y^2 - z^2 = 1$

IV



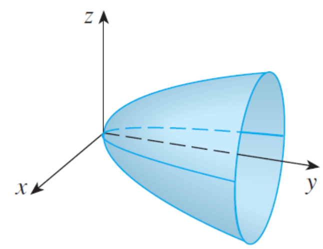
(B) $9x^2 + y^2 + z^2 = 1$

V



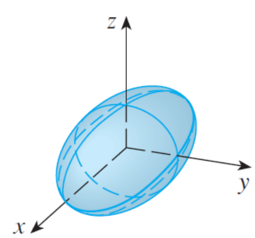
(H) $y = x^2 - z^2$

VI



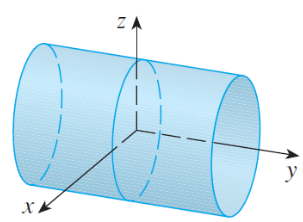
(E) $y = 2x^2 + z^2$

VII



(A) $x^2 + 4y^2 + 9z^2 = 1$

VIII



(G) $x^2 + 2z^2 = 1$

6. Describe the three dimensional object defined by each equation in spherical coordinates.

(A) $\rho = R$ (C) $\phi = \pi/2$ (E) $\phi = C$ for $C \neq 0, \dfrac{\pi}{2}, \pi$

(B) $\phi = 0$ (D) $\phi = \pi$

(A) The equation $\rho = R$ defines a sphere of radius R.

(B) Points with $\phi = 0$ are on the positive z-axis.

(C) Points with $\phi = \pi/2$ are on the xy-plane (i.e., $z = 0$).

(D) Points with $\phi = \pi$ are on the negative z-axis.

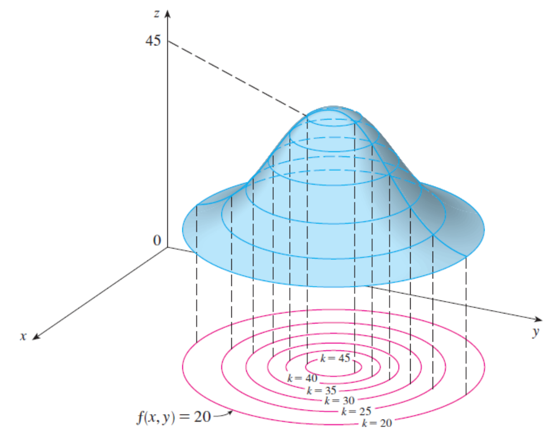
(E) For other values of C, the equation $\phi = C$ defines a nappe of a cone.

Multivariable Functions: (Sections 14.1 - 14.5)

2-variable functions $z = f(x, y)$ have domain in $\mathbb{R}^2$ and range in$\mathbb{R}$. The graph of f is a surface S in $\mathbb{R}^3$, a 2-dimensional object in space. The graph of S can be approximated with curves in two ways:

(i) Horizontal and Vertical Traces:
A curve obtained by intersecting the graph with a plane parallel to the major planes.

- The trace of $x = a$ results in $z = f(a, y)$.
- The trace of $y = b$ results in $z = f(x, b)$.
- The trace of $z = c$ results in $c = f(x, y)$.



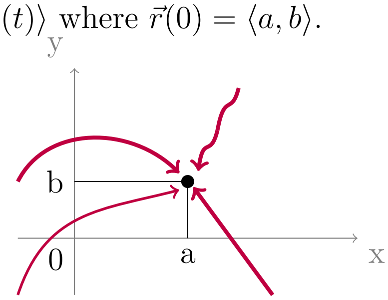
(ii) Level Curves and Contour Maps:

A level curve is a curve $f(x, y) = c$ in $\mathbb{R}^2$, where c is a constant. A contour map is a collection of level curves of various constants.

Limits in 2-Variables:

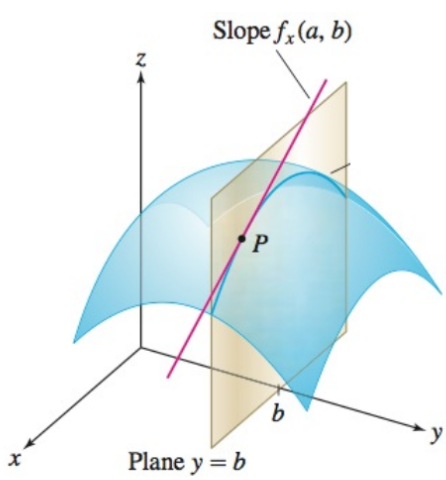
$\lim_{(x,y)\to(a,b)} f(x, y) = L$ if and only if $\lim_{t\to 0} f(x(t), y(t)) = L$ along any curve $\vec{r}(t) = \langle x(t), y(t)\rangle$ where $\vec{r}(0) = \langle a, b\rangle$.

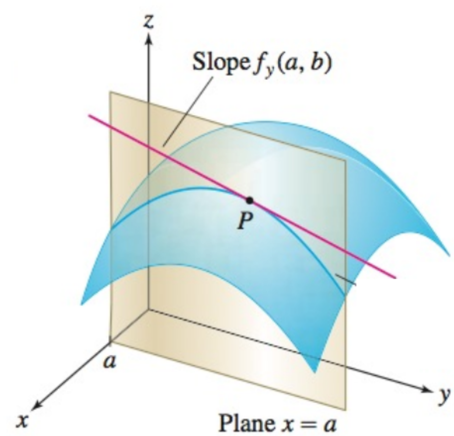
- Show a limit does not exist by exhibiting two paths with different limits.
- Show a limit exists by using substitution (continuous values only), using the Squeeze Theorem, or using Polar Coordinates.



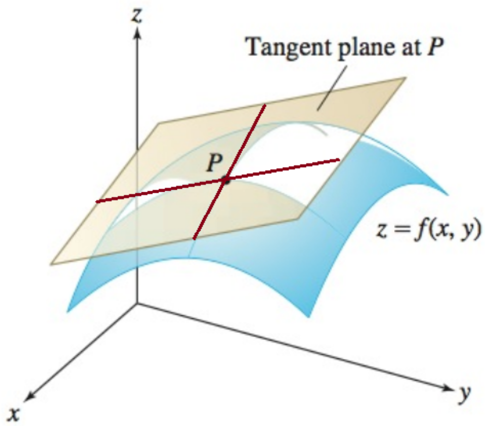
Partial Derivatives and Tangent Planes:

The rate of change in the x-direction is the partial derivative $f_x = \dfrac{\partial f}{\partial x}$. The rate of change in the y-direction is the partial derivative $f_y = \dfrac{\partial f}{\partial y}$. Note that $f_{xy} = \dfrac{\partial^2 f}{\partial y \partial x}$.





To compute the partial derivative with respect to x treat the y-variable as a constant and apply the ordinary rules for differentiation. Compute the partial derivative with respect to y in an analogous way. Partial differentiation is **not** implicit differentiation.



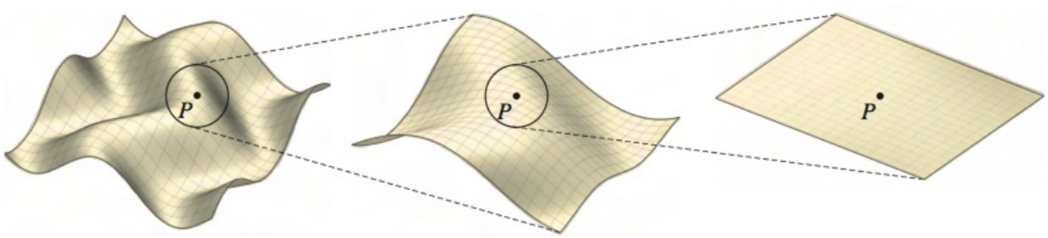
The **tangent plane** to the surface S, defined by $z = f(x, y)$, at $P(a, b)$ contains the tangent line of any curve on S which passes through P.

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

f is differentiable at P if it is locally linear at P.

If $f_x(a, b)$ and $f_y(a, b)$ are continuous, then f is differentiable at (a, b).

If f is differentiable at (a, b), then the tangent plane to f at (a, b) approximates values of f for points near (a, b).



If $f(x_1, x_2, \ldots, x_n)$ is differentiable $(a_1, a_2, \ldots, a_n)$, then the total differential df approximates changes in f over small changes in the domain away from $(a_1, a_2, \ldots, a_n)$.

$$df = f_{x_1}(a_1, \ldots, a_n)\Delta x_1 + f_{x_2}(a_1, \ldots, a_n)\Delta x_2 + \ldots + f_{x_n}(a_1, \ldots, a_n)\Delta x_n$$

If f is a differentiable function, then f has a directional derivative in the direction of any unit vector $\vec{u} = \langle a, b \rangle$ and

$$D_{\vec{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b = \nabla f(x, y) \cdot \vec{u}$$

Basic geometric properties of the gradient vector:

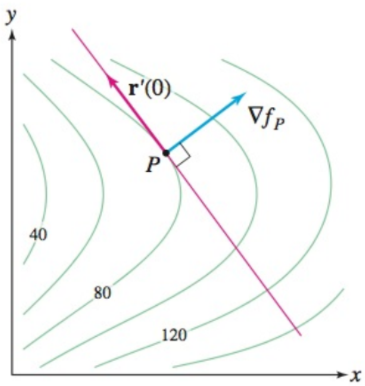
(i) $\nabla f(a, b)$ points in the direction of maximum rate of increase of f at (a, b). The maximum rate of increase is $|\nabla f(a, b)|$.

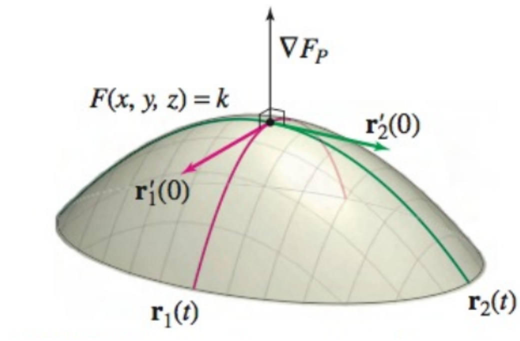
(ii) $-\nabla f(a, b)$ points in the direction of maximum rate of decrease of f at (a, b). The maximum rate of decrease is $-|\nabla f(a, b)|$.

(iii) $\nabla f(a, b)$ is orthogonal to the level curve through (a, b).

(iv) $\nabla f(a, b, c)$ is orthogonal to the level surface through (a, b, c).

(v) The equation of the tangent plane to the level surface $F(x, y, z) = k$ at (a, b, c) has a normal vector $\nabla f(a, b, c)$.





Exercises:

1. Evaluate the following limits:

(A) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2+3xy+4y^2}{3x^2+5y^2}$

Path $y=0$**:** $\displaystyle\lim_{(x,0)\to(0,0)} \frac{2x^2}{3x^2} = \frac{2}{3}$ **Path** $x=0$**:** $\displaystyle\lim_{(0,y)\to(0,0)} \frac{4y^2}{5y^2} = \frac{4}{5}$

The limit does not exist.

(B) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4x^2y}{x^2+y^2}$

Solution 1:
Using polar coordinates:

$$\lim_{(x,y)\to(0,0)} \frac{4x^2y}{x^2+y^2} = \lim_{r\to0^+} \frac{4r^2\cos^2(\theta)r\sin(\theta)}{r^2} = \lim_{r\to0^+} 4r\cos^2(\theta)\sin(\theta) = 0$$

Solutions 2:
$0 \le \dfrac{4x^2y}{x^2+y^2} \le 4y$ since $0 \le \dfrac{1}{x^2+y^2} \le \dfrac{1}{x^2}$

Since $\displaystyle\lim_{(x,y)\to(0,0)} 4y = 0$, by the Squeeze Theorem $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4x^2y}{x^2+y^2} = 0$.

(C) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$

Using polar coordinates:

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to0^+} \frac{r^3\left(\cos^3(\theta)+\sin^3(\theta)\right)}{r^2} = \lim_{r\to0^+} r\left(\cos^3(\theta)+\sin^3(\theta)\right) = 0$$

(D) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2+y^2}{x^2+2y^2}$

Path $y = 0$**:** $\lim_{(x,0)\to(0,0)} \frac{2x^2}{x^2} = 2$ **Path** $x = 0$**:** $\lim_{(0,y)\to(0,0)} \frac{y^2}{2y^2} = \frac{1}{2}$

The limit does not exist.

(E) $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2 + y^8}$

Path $y = 0$**:** $\lim_{(x,0)\to(0,0)} \frac{0}{x^2} = 0$ **Path** $x = y^4$**:** $\lim_{(y^4,y)\to(0,0)} \frac{y^8}{y^8 + y^8} = \frac{1}{2}$

The limit does not exist.

(F) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + xy + y^2}$

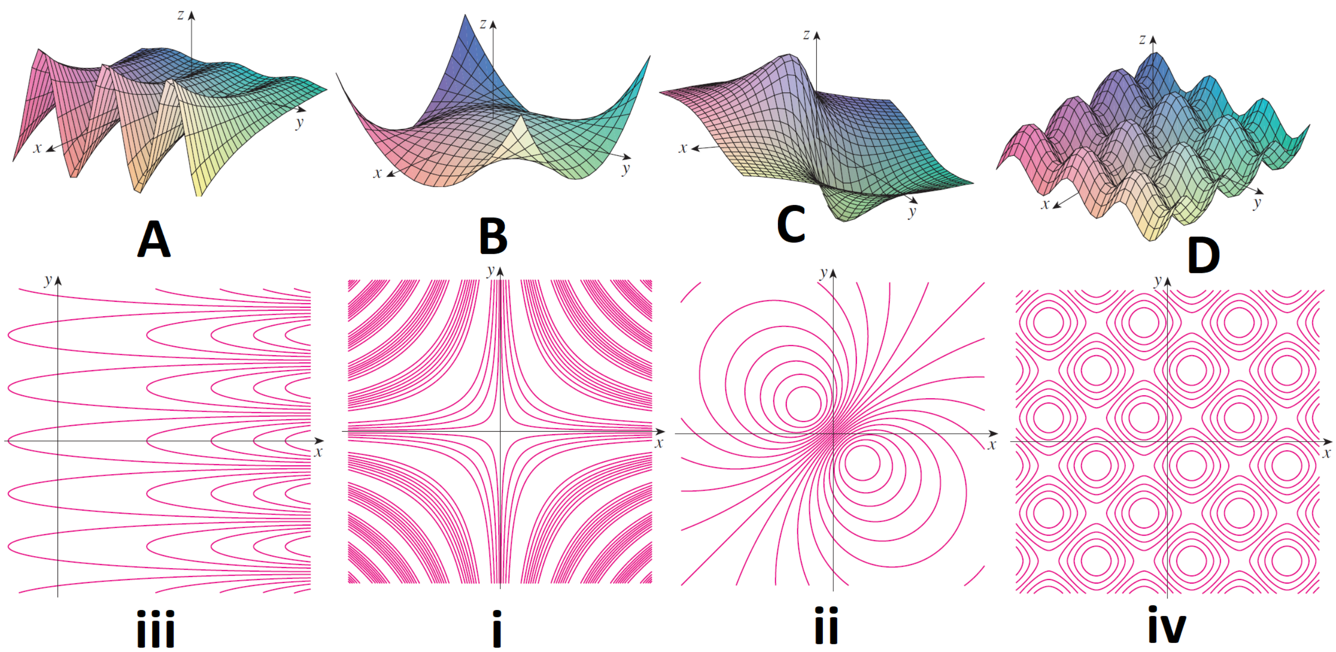
Path $x = 0$**:** $\lim_{(0,y)\to(0,0)} \frac{0}{y^2} = 0$

Path $y = mx$**:**

$$\lim_{(x,mx)\to(0,0)} \frac{m^2x^2}{x^2 + mx^2 + m^2x^2} = \frac{m^2}{1 + m + m^2}$$

The limit does not exist.

2. Match the surface and contour map:



3. Compute $\frac{\partial^2 h}{\partial y\, \partial x}$ for $h(x, y) = \ln(x^3 + y^3)$

$$\frac{\partial^2 h}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\ln(x^3+y^3)\right) = \frac{\partial}{\partial y}\left(\frac{3x^2}{x^3+y^3}\right) = \frac{-9x^2y^2}{(x^3+y^3)^2}$$

4. A one-meter long bar is heated unevenly, with the temperature in $^\circ C$ at a distance x from one end at time t given by

$$H(x,t) = 100e^{-0.1t}\sin(\pi x) \quad 0 \le x \le 1$$

(A) Calculate $H_x(0.2,t)$ and $H_x(0.8,t)$. What is the practical interpretation (in terms of temperature) of these two partial derivatives? Explain why each one has the sign that it does.

(B) Calculate $H_t(x,t)$. What is its sign? What is its interpretation in terms of temperature?

$$H_x(x,t) = 100\pi e^{-0.1t}\cos(\pi x) \qquad H_t(x,t) = -10e^{-0.1t}\sin(\pi x)$$

(A) $H_x(0.2,t) \approx 254e^{-0.1t}$ and $H_x(0.8,t) \approx -254e^{-0.1t}$.

0.2 meters from the bottom of the bar at time t, the instantaneous change in temperature as we move up the bar decreases by approximately $254e^{-0.1t}$ degrees per meter moved.

0.8 meters from the bottom of the bar at time t, the instantaneous change in temperature as we move up the bar increases by approximately $254e^{-0.1t}$ degrees per meter moved.

A possible scenario is the center of the bar was heated and the ends at $x = 0$ and $x = 1$ are cold.

(B) $H_t(x,t)$ is negative. As time passes, the bar cools.

5. Is there a function f which has the following partial derivatives? If so, what is it? Are there any others?

$$f_x(x,y) = 4x^3y^2 - 3y^4 \qquad f_y(x,y) = 2x^4y - 12xy^3$$

$$f_{xy}(x,y) = 8x^3y - 12y^3 \qquad f_{yx}(x,y) = 8x^3y - 12y^3$$

Since f_x and f_y are continuous everywhere, Clairaut's Theorem indicates that $f_{xy} = f_{yx}$ everywhere; such a function f might exist.

The function could be $f(x,y) = x^4y^2 - 3xy^4 + C$ where C is a constant.

6. If $|a|$ is much greater than $|b|, |c|, |d|$, to which of a, b, c, d is the value of the determinant most sensitive? Justify your answer.

$$f(a,b,c,d) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$f(a,b,c,d) = ad - bc \qquad \nabla f(a,b,c,d) = \langle d, -c, -b, a\rangle$$

Using the Total Differential, $df = d\Delta a - c\Delta b - b\Delta c + a\Delta d$.

Since $|a|$ is much greater than $|b|, |c|, |d|$, the error in calculating the determinant, df, will be most sensitive to errors in measuring d.

7. Four positive numbers, each less than 50, are rounded to the second decimal place and than multiplied together. Use differentials to estimate the maximum possible errors in the computed product that might result from the rounding.

$$P(x_1,x_2,x_3,x_4) = x_1x_2x_3x_4 \qquad \nabla P(x_1,x_2,x_3,x_4) = \langle x_2x_3x_4,\ x_1x_3x_4,\ x_1x_2x_4,\ x_1x_2x_3\rangle$$

Using the Total Differential, $dP = x_2x_3x_4\Delta x_1 + x_1x_3x_4\Delta x_2 + x_1x_2x_4\Delta x_3 + x_1x_2x_3\Delta x_4$.

Since $x_i < 50$ and $|\Delta x_i| < 0.01$, the error in calculating the product is smaller than

$$|dP| < (50)^3(0.01) + (50)^3(0.01) + (50)^3(0.01) + (50)^3(0.01) = 5000$$

8. Verify that $f(x,y) = \sqrt{y + \cos^2(x)}$ is differentiable at $(0,0)$ and then show that

$$\sqrt{y+\cos^2(x)} \approx 1 + 0.5y$$

$$f(0,0) = 1 \qquad f_x(x,y) = \frac{-\cos(x)\sin(x)}{\sqrt{y+\cos^2(x)}} \qquad f_y(x,y) = \frac{1}{2\sqrt{y+\cos^2(x)}}$$

Since $f_x(0,0) = 0$ and $f_y(0,0) = \dfrac{1}{2}$ and both f_x and f_y are elementary, they are continuous at $(0,0)$. Since f_x and f_y are continuous at $(0,0)$, f is differentiable at $(0,0)$.

Since f is differentiable at $(0,0)$, f can be approximated by its' tangent plane at $(0,0)$ for (x,y) near $(0,0)$.

$$f(x,y) \approx L_{(0,0)}(x,y) = 1 + 0(x-0) + 0.5(y-0) = 1 + 0.5y$$

9. Use a linear approximation to estimate $\dfrac{9.02}{\sqrt{2.01\cdot 1.99}}$

Let $f(x,y,z) = \dfrac{x}{\sqrt{yz}}$.

$$f_x(x,y,z) = \frac{1}{\sqrt{yz}} \qquad f_y(x,y,z) = \frac{-xz}{2(yz)^{3/2}} \qquad f_z(x,y,z) = \frac{-xy}{2(yz)^{3/2}}$$

Since f_x, f_y, and f_z are continuous at $(9,2,2)$, f is differentiable at $(9,2,2)$.

$$f(9,2,2) = \frac{9}{2} \qquad f_x(9,2,2) = \frac{1}{2} \qquad f_y(9,2,2) = \frac{-9}{8} \qquad f_z(9,2,2) = \frac{-9}{8}$$

$$L_{(9,2,2)}(x,y,z) = \frac{9}{2} + \frac{1}{2}(x-9) - \frac{9}{8}(y-2) - \frac{9}{8}(z-2)$$

Since f is differentiable at $(9,2,2)$,

$$\frac{9.02}{\sqrt{2.01 \cdot 1.99}} \approx L_{(9,2,2)}(9.02, 2.01, 1.99) = \frac{9}{2} + \frac{1}{2}(0.02) - \frac{9}{8}(0.01) - \frac{9}{8}(-0.01) = 4.51$$

10. Calculate the directional derivative of $f(x,y,z) = 3e^x \cos(yz)$ in the direction $\langle 2, 1, -2 \rangle$.

$$f_x(x,y,z) = 3e^x \cos(yz) \qquad f_y(x,y,z) = -3ze^x \sin(yz) \qquad f_z(x,y,z) = -3ye^x \sin(yz)$$

The direction is $\vec{u} = \left\langle \frac{2}{3}, \frac{1}{3}, \frac{-2}{3} \right\rangle$. Since f is differentiable everywhere,

$$D_{\vec{u}}(x,y,z) = \nabla f(x,y,z) \cdot \vec{u} = e^x \big(2\cos(yz) - z\sin(yz) + 2y\sin(yz)\big)$$

11. A bug located at $(3,9,4)$ begins traveling in a straight line towards $(5,7,3)$. The temperature is

$$T(x,y,z) = xe^{y-z}$$

where x, y, z are in meters and T is in $°C$.

(A) At what rate is the bug's temperature changing?

$$T_x(x,y,z) = e^{y-z} \qquad T_y(x,y,z) = xe^{y-z} \qquad T_z(x,y,z) = -xe^{y-z}$$

The bug travels in the direction of $\vec{u} = \left\langle \frac{2}{3}, \frac{-2}{3}, \frac{-1}{3} \right\rangle$. Since T is differentiable,

$$D_{\vec{u}}f(3,9,4) = \nabla T(3,9,4) \cdot \vec{u} = \left\langle e^5, 3e^5, -3e^5 \right\rangle \cdot \left\langle \frac{2}{3}, \frac{-2}{3}, \frac{-1}{3} \right\rangle = \frac{-e^5}{3} \frac{°C}{m}$$

(B) At $(5,7,3)$, in what direction is the temperature increasing the fastest? What is this maximum rate of change?

The temperature is increasing fastest in the direction $\dfrac{\nabla T(5,7,3)}{|\nabla T(5,7,3)|}$ at a rate of $|\nabla T(5,7,3)|$.

$$\nabla T(5,7,3) = \left\langle e^4, 5e^4, -5e^4 \right\rangle \qquad |\nabla T(5,7,3)| = \sqrt{51}e^4$$

$$\frac{\nabla T(5,7,3)}{|\nabla T(5,7,3)|} = \left\langle \frac{1}{\sqrt{51}}, \frac{5}{\sqrt{51}}, \frac{-5}{\sqrt{51}} \right\rangle$$

(C) At $(5,7,3)$, in what direction is the temperature decreasing the fastest? What is this maximum rate of change?

The temperature is decreasing fastest in the direction $\dfrac{-\nabla T(5,7,3)}{|\nabla T(5,7,3)|}$ at a rate of $-|\nabla T(5,7,3)|$.

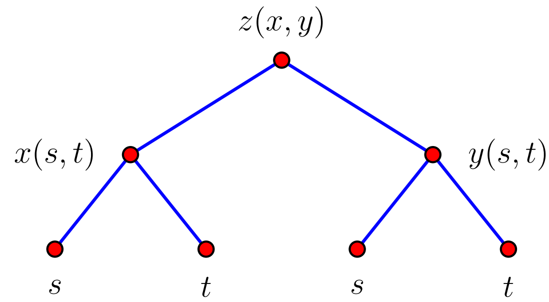
$$-|\nabla T(5,7,3)| = -\sqrt{51}e^4 \qquad \frac{-\nabla T(5,7,3)}{|\nabla T(5,7,3)|} = \left\langle \frac{-1}{\sqrt{51}}, \frac{-5}{\sqrt{51}}, \frac{5}{\sqrt{51}} \right\rangle$$

Chain Rule and Optimization: (Sections 14.6, 14.7, 14.8)

Chain Rule: Partial derivatives through composition of multiple variable functions.

Suppose that $z = f(x,y)$, $x = g(s,t)$, and $y = h(s,t)$ are differential functions. Then z can be viewed as a function of s and t which is differentiable where

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$



In general, if $f(x_1, \ldots, x_n)$ is a function of n-variables which can be shown to depend upon m-independent variables $\{t_1, \ldots, t_m\}$ through k-composition connections, then

- f has m-partial derivatives,
- Each partial derivative is a sum with n terms,
- Each term of the sum is a product of $(k+1)$ terms.

Implicit Differentiation: If $z = z(x,y)$ is defined implicitly by $F(x,y,z) = 0$, by the Chain Rule

$$\frac{\partial z}{\partial x} = \frac{-F_x(x,y,z)}{F_z(x,y,z)} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = \frac{-F_y(x,y,z)}{F_z(x,y,z)}$$

Critical points of f are the points in the domain where $\nabla f = \vec{0}$. *Fermat's Theorem* states that the critical points of f are the only potential local extrema of f.

Optimization - Local Extrema: *Critical points* of f are the points in the domain where $\nabla f = \vec{0}$. *Fermat's Theorem* states that the critical points of f are the only potential local extrema of f.

$$\textbf{Discriminant: } D(a,b) = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$$

Second Derivative Test: If (a,b) is a critical point of f and the second-order partial derivatives of f are continuous near (a,b), then

(I) If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then (a,b) is a local minimum of f.

(II) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then (a,b) is a local maximum of f.

(III) If $D(a,b) < 0$, then (a,b) is a saddle point.

Lagrange Multipliers: If f and g are differentiable functions and f has a local extrema on the constraint curve $g(x,y) = k$ at (a,b), where $\nabla g(a,b) \neq \vec{0}$, then there exists a scalar λ such that $\nabla f(a,b) = \lambda \nabla g(a,b)$. [1]

Optimization - Absolute Extrema: The *Extreme Value Theorem* guarantees that functions which are continuous on a *closed* and *bounded* set $\mathcal{D}$ attain an absolute maximum and minimum value in $\mathcal{D}$.

Absolute extrema of a continuous function on a closed and bounded set are located using the *Closed Interval Method*.

[1] The Lagrange Multipliers are not included in Exam 1.

(I) Find all critical points in $\mathcal{D}$ and their values.

(II) Find the values of the absolute extrema of f on the boundary of $\mathcal{D}$ using either

(a) substitution and the closed interval method from MATH 125,

or (b) Lagrange Multipliers.

(III) The largest values from (I) and (II) are the absolute maximum values and the smallest values are the absolute minimum values.

Exercises:

1. Find $\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}, \frac{\partial z}{\partial w}$ when $(u, v, w) = (2, 1, 0)$.

$$z = x^2 + xy^3 \qquad x = uv^2 + w^3 \qquad y = u + ve^w$$

When $(u, v, w) = (2, 1, 0)$, $(x, y) = (2, 3)$. Using the Chain Rule,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \left(2x + y^3\right)\left(v^2\right) + (3y^2)(1) = 58$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \left(2x + y^3\right)(2uv) + (3y^2)(e^w) = 151$$

$$\frac{\partial z}{\partial w} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial w} = \left(2x + y^3\right)(3w^2) + (3y^2)(ve^w) = 54$$

2. Consider the implicit surface:

$$xyz = \cos(x + y + z)$$

(A) Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

(B) Find the equation of tangent plane at $\left(\frac{\pi}{2}, 0, 0\right)$.

(A) Let $F(x, y, z) = xyz - \cos(x + y + z)$. z is implicitly defined by $F(x, y, z) = 0$. Using the Chain Rule,

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-\left(yz + \sin(x + y + z)\right)}{xy + \sin(x + y + z)} \qquad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-\left(xz + \sin(x + y + z)\right)}{xy + \sin(x + y + z)}$$

(B) **Method 1: (Using the partials)**

Use the formula: $z = c + \frac{\partial z}{\partial x}(a, b, c)(x - a) + \frac{\partial z}{\partial x}(a, b, c)(y - b)$.

That is, $z = \frac{\partial z}{\partial x}\left(\frac{\pi}{2}, 0, 0\right)\left(x - \frac{\pi}{2}\right) + \frac{\partial z}{\partial x}\left(\frac{\pi}{2}, 0, 0\right)(y)$: $\boxed{z = -\left(x - \frac{\pi}{2}\right) - y}$

Method 2: (Using the gradient)

$\nabla F(x,y,z) = \langle yz + \sin(x+y+z),\ xz + \sin(x+y+z),\ xy + \sin(x+y+z)\rangle$

$\nabla F\left(\frac{\pi}{2},\ 0,\ 0\right) = \langle 1,\ 1,\ 1\rangle$ An equation of the plane is: $\langle 1,\ 1,\ 1\rangle \cdot \left\langle x - \frac{\pi}{2},\ y,\ z\right\rangle = 0$

So the equation is $\boxed{x - \dfrac{\pi}{2} + y + z = 0}$

3. For the following functions, find the critical points and classify them using the second derivative test:

(A) $f(x,y) = 2xy - x^2 - 2y^2 + 3x + 4$ (B) $g(x,y) = e^{x^2+y^2-4x}$

(A) The critical points of f satisfy the equation $\nabla f(x,y) = \vec{0}$.

$$\nabla f(x,y) = \langle 2y - 2x + 3, 2x - 4y\rangle = \vec{0} \qquad \Rightarrow \qquad (x,y) = \left(3, \frac{3}{2}\right)$$

$$D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - f_{xy}(x,y)^2 = (-2)(-4) - (2)^2 = 4$$

By the Second Derivative Test, $\left(3, \frac{3}{2}\right)$ is a local maximum of f since

$$D\left(3, \frac{3}{2}\right) = 4 \qquad f_{xx}\left(3, \frac{3}{2}\right) = -2$$

(B) The critical points of g satisfy the equation $\nabla g(x,y) = \vec{0}$.

$$\nabla g(x,y) = \left\langle (2x-4)e^{x^2+y^2-4x}, 2ye^{x^2+y^2-4x}\right\rangle = \vec{0} \qquad \Rightarrow \qquad (x,y) = (2,0)$$

$$D(x,y) = \left((4x^2 - 16x + 18)e^{x^2+y^2-4x}\right)\left((2+4y^2)e^{x^2+y^2-4x}\right) - \left(2y(2x-4)e^{x^2+y^2-4x}\right)^2$$

By the Second Derivative Test, $(2,0)$ is a local minimum of g since

$$D(2,0) = 4e^{-8} \qquad g_{xx}(2,0) = 2e^{-4}$$